CE 297: Problems in the Mathematical Theory of Elasticity: Homework II

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In the following, the positive sense (counterclockwise traversal) is to be assumed when integrating over closed contours unless mentioned otherwise. A (*) indicates somewhat harder problems.

1. Select any branch of the multifunction $f(z) = \sqrt{2z + z_0}$. Show that this branch is holomorphic in the cut plane and find its derivative.

2. Find the branch points of the multifunction $f(z) = \log(z^2 - 1)$. What are the potential branch cuts for this function and how are they specified? Calculate the derivative of these branches in the cut plane.

3. Repeat the exercise for the multifunction $f(z) = \log\left(\frac{z+1}{z-1}\right)$.

4. For the function $\sqrt{(z-a)(z-b)}$ find the branch cuts corresponding to the choice $-\pi \leq \theta_1 < \pi$, $-\pi \leq \theta_2 < \pi$, where θ_1 and θ_2 are angular coordinates in the double polar coordinate system

$$z - a = r_2 \exp(i\theta_2) \qquad z - b = r_1 \exp(i\theta_1)$$

   Assume that a and b are real numbers and $a < b$.

5. Use parametrization to evaluate the integral

$$\mathcal{I}_n = \oint_C z^n \, dz \quad n = 0, \pm 1, \pm 2, \cdots$$

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where C is the unit circle $|z| = 1$ (traversed CCW).

For which values of n can you use the Fundamental theorem / anti-derivative to evaluate $\mathcal{I}_n$?

6. * Use the principal branches of the logarithm and square-root functions to evaluate the integrals
$$\int_{-a}^{a} \log z \, dz \qquad \int_{-a}^{a} \sqrt{z} \, dz$$
where a is a positive real number. Explain your reasoning.

7. Consider the integral
$$I = \int_C \frac{\exp(i\, z)}{z^2} \, dz$$
where C is the semi-circle with endpoints $(R, 0)$ and $(-R, 0)$ in the upper half-plane. Show that
$$\lim_{R \to \infty} I = 0$$

8. Evaluate the integral
$$I = \oint_C \frac{\exp(z^2)}{z^2} \, dz$$
where C is any simple closed curve enclosing the origin.

9. Recall that if $f(z)$ is a holomorphic function in the interior D of a simple closed contour C, then all the derivatives $f^{(k)}(z)$ exist in D and are given by
$$f^{(k)}(z) = \frac{k!}{2\pi i} \oint_C \frac{f(\zeta)}{(\zeta - z)^{k+1}} \, d\zeta$$
Show that this formula can be 'derived' (a rigorous proof is not needed) by iteratively applying the limit definition of a derivative. State any extra assumptions used.

10. Using parametrization, you know that if C is the unit circle $|z| = 1$, the integral
$$\oint_C \frac{1}{z^2} \, dz = 0$$
Show that this integral is also 0 for all other simple closed contours $C^* \neq C$ enclosing the origin, i.e. there is no C^* for which $\oint_{C^*} \frac{1}{z^2} \, dz$ is *not* zero. Explain why this result does not contradict Morera's theorem.

11. Re-derive the Cauchy integration formula for a function f which is holomorphic in the exterior of a Jordan contour $\mathcal{C}$, i.e. in $S^- \cup \mathbb{C}_\infty$ where the point z lies in S^-.

12. Consider the function $\exp(1/z)$. Evaluate the integral

$$\oint_C \frac{\exp(1/t)}{t-z}\,dt$$

where C is the unit circle $|t| = 1$ (traversed CCW) and z is any point lying *outside* the unit circle.

13. Find the Taylor series of the following functions about $z_0 = 0$ and their radii of convergence

$$\frac{z}{1+z^2} \qquad \cosh z \qquad \frac{\cos(z)-1}{z^2}$$

14. Find the Taylor series of the function $1/(1+z)$ about the points $z_0 = -5i$, $z_0 = 2+3i$. Can you estimate the radii of convergence of these Taylor series *without* writing them down explicitly?

15. * Consider the sequence of Euler numbers E_n which are defined as coefficients appearing in the power series

$$\frac{1}{\cosh\, z} = \sum_{n=0}^{\infty} \frac{E_n}{n!} z^n$$

Find the radius of convergence of this series and the first six Euler numbers E_n.